This is a submission to one of journals of TMRG: BJMA/AFA

EXTENSION OF THE REFINED JENSEN'S OPERATOR INEQUALITY WITH CONDITION ON SPECTRA

JADRANKA MIĆIĆ[1]*, JOSIP PEČARIĆ[2] AND JURICA PERIĆ[3]

Abstract. We give an extension of the refined Jensen's operator inequality for $n-$tuples of self-adjoint operators, unital $n-$tuples of positive linear mappings and real valued continuous convex functions with conditions on the spectra of the operators. We also study the order among quasi-arithmetic means under similar conditions.

1. Introduction

We recall some notations and definitions. Let $\mathcal{B}(H)$ be the C^*-algebra of all bounded linear operators on a Hilbert space H and 1_H stands for the identity operator. We define bounds of a self-adjoint operator $A \in \mathcal{B}(H)$ by

$$m_A = \inf_{\|x\|=1} \langle Ax, x\rangle \quad \text{and} \quad M_A = \sup_{\|x\|=1} \langle Ax, x\rangle$$

for $x \in H$. If $\mathsf{Sp}(A)$ denotes the spectrum of A, then $\mathsf{Sp}(A)$ is real and $\mathsf{Sp}(A) \subseteq [m_A, M_A]$.

For an operator $A \in \mathcal{B}(H)$ we define operators $|A|$, A^+, A^- by

$$|A| = (A^*A)^{1/2}, \qquad A^+ = (|A| + A)/2, \qquad A^- = (|A| - A)/2.$$

Obviously, if A is self-adjoint, then $|A| = (A^2)^{1/2}$ and $A^+, A^- \geq 0$ (called positive and negative parts of $A = A^+ - A^-$).

B. Mond and J. Pečarić in [9] proved Jensen's operator inequality

$$f\left(\sum_{i=1}^n w_i \Phi_i(A_i)\right) \leq \sum_{i=1}^n w_i \Phi_i\left(f(A_i)\right), \tag{1.1}$$

for operator convex functions f defined on an interval I, where $\Phi_i : \mathcal{B}(H) \to \mathcal{B}(K)$, $i = 1, \ldots, n$, are unital positive linear mappings, $A_1, \ldots, A_n$ are self-adjoint operators with the spectra in I and $w_1, \ldots, w_n$ are non-negative real numbers with $\sum_{i=1}^n w_i = 1$.

Date: Received: xxxxxx; Revised: yyyyyy; Accepted: zzzzzz.

* Corresponding author.

2010 *Mathematics Subject Classification.* Primary 47A63; Secondary 47B15.

Key words and phrases. Self-adjoint operator, positive linear mapping, convex function, Jensen's operator inequality, quasi-arithmetic mean.

F. Hansen, J. Pečarić and I. Perić gave in [3] a generalization of (1.1) for a unital field of positive linear mappings. The following discrete version of their inequality holds

$$f\left(\sum_{i=1}^{n}\Phi_i(A_i)\right)\leq\sum_{i=1}^{n}\Phi_i\left(f(A_i)\right), \tag{1.2}$$

for operator convex functions f defined on an interval I, where $\Phi_i:\mathcal{B}(H)\to\mathcal{B}(K)$, $i=1,\dots,n$, is a unital field of positive linear mappings (i.e. $\sum_{i=1}^{n}\Phi_i(1_H)=1_K$), $A_1,\dots,A_n$ are self-adjoint operators with the spectra in I.

Recently, J. Mićić, Z. Pavić and J. Pečarić proved in [5, Theorem 1] that (1.2) stands without operator convexity of $f:I\to\mathbb{R}$ if a condition on spectra

$$(m_A,M_A)\cap[m_i,M_i]=\emptyset\quad\text{for } i=1,\dots,n$$

holds, where m_i and M_i, $m_i\leq M_i$ are bounds of A_i, $i=1,\dots,n$; and m_A and M_A, $m_A\leq M_A$, are bounds of $A=\sum_{i=1}^{n}\Phi_i(A_i)$ (provided that the interval I contains all m_i,M_i).

Next, they considered in [6, Theorem 2.1] the case when $(m_A,M_A)\cap[m_i,M_i]=\emptyset$ is valid for several $i\in\{1,\dots,n\}$, but not for all $i=1,\dots,n$ and obtain an extension of (1.2) as follows.

Theorem A. Let $(A_1,\dots,A_n)$ be an $n-$tuple of self-adjoint operators $A_i\in B(H)$ with the bounds m_i and M_i, $m_i\leq M_i$, $i=1,\dots,n$. Let $(\Phi_1,\dots,\Phi_n)$ be an $n-$tuple of positive linear mappings $\Phi_i:B(H)\to B(K)$, such that $\sum_{i=1}^{n_1}\Phi_i(1_H)=\alpha\,1_K$, $\sum_{i=n_1+1}^{n}\Phi_i(1_H)=\beta\,1_K$, where $1\leq n_1<n$, $\alpha,\beta>0$ and $\alpha+\beta=1$. Let $m=\min\{m_1,\dots,m_{n_1}\}$ and $M=\max\{M_1,\dots,M_{n_1}\}$. If

$$(m,M)\cap[m_i,M_i]=\emptyset\qquad\text{for}\quad i=n_1+1,\dots,n,$$

and one of two equalities

$$\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(A_i)=\sum_{i=1}^{n}\Phi_i(A_i)=\frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(A_i)$$

is valid, then

$$\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f(A_i))\leq\sum_{i=1}^{n}\Phi_i(f(A_i))\leq\frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(f(A_i)), \tag{1.3}$$

holds for every continuous convex function $f:I\to\mathbb{R}$ provided that the interval I contains all m_i,M_i, $i=1,\dots,n$,.

If $f:I\to\mathbb{R}$ is concave, then the reverse inequality is valid in (1.3).

Very recently, J. Mićić, J. Pečarić and J. Perić gave in [7, Theorem 3] the following refinement of (1.2) with condition on spectra, i.e. a refinement of [5, Theorem 3] (see also [5, Corollary 5]).

Theorem B. Let $(A_1,\dots,A_n)$ be an $n-$tuple of self-adjoint operators $A_i\in B(H)$ with the bounds m_i and M_i, $m_i\leq M_i$, $i=1,\dots,n$. Let $(\Phi_1,\dots,\Phi_n)$ be an $n-$tuple of positive linear mappings $\Phi_i:B(H)\to B(K)$, $i=1,\dots,n$, such that $\sum_{i=1}^{n}\Phi_i(1_H)=1_K$. Let

$$(m_A,M_A)\cap[m_i,M_i]=\emptyset\quad\text{for } i=1,\dots,n,\qquad\text{and}\qquad m<M,$$

where m_A and M_A, $m_A \leq M_A$, are the bounds of the operator $A = \sum_{i=1}^n \Phi_i(A_i)$ and

$$m = \max\{M_i \colon M_i \leq m_A, i \in \{1,\dots,n\}\}, \ M = \min\{m_i \colon m_i \geq M_A, i \in \{1,\dots,n\}\}.$$

If $f : I \to \mathbb{R}$ is a continuous convex (resp. concave) function provided that the interval I contains all m_i, M_i, then

$$f\left(\sum_{i=1}^n \Phi_i(A_i)\right) \leq \sum_{i=1}^n \Phi_i\left(f(A_i)\right) - \delta_f \widetilde{A} \leq \sum_{i=1}^n \Phi_i\left(f(A_i)\right) \tag{1.4}$$

$$\text{(resp.} \quad f\left(\sum_{i=1}^n \Phi_i(A_i)\right) \geq \sum_{i=1}^n \Phi_i\left(f(A_i)\right) + \delta_f \widetilde{A} \geq \sum_{i=1}^n \Phi_i\left(f(A_i)\right)\ )$$

holds, where

$$\begin{aligned} \delta_f \equiv \delta_f(\bar{m},\bar{M}) &= f(\bar{m}) + f(\bar{M}) - 2f\left(\tfrac{\bar{m}+\bar{M}}{2}\right) \\ \text{(resp.} \quad \delta_f \equiv \delta_f(\bar{m},\bar{M}) &= 2f\left(\tfrac{\bar{m}+\bar{M}}{2}\right) - f(\bar{m}) - f(\bar{M})\ ), \\ \widetilde{A} \equiv \widetilde{A}_A(\bar{m},\bar{M}) &= \tfrac{1}{2}1_K - \tfrac{1}{\bar{M}-\bar{m}}\left|A - \tfrac{\bar{m}+\bar{M}}{2}1_K\right| \end{aligned}$$

and $\quad \bar{m} \in [m, m_A]$, $\bar{M} \in [M_A, M]$, $\bar{m} < \bar{M}$, $\quad$ are arbitrary numbers.

There is an extensive literature devoted to Jensens inequality concerning different refinements and extensive results, see, for example [1], [2], [4], [10]–[14].

In this paper we study an extension of Jensen's inequality given in Theorem B and a refinement of Theorem A. As an application of this result to the quasi-arithmetic mean with a weight, we give an extension of results given in [7] and a refinement of ones given in [6].

2. Main results

To obtain our main result we need a result [7, Lemma 2] given in the following lemma.

Lemma C. Let A be a self-adjoint operator $A \in B(H)$ with $\mathsf{Sp}(A) \subseteq [m, M]$, for some scalars $m < M$. Then

$$\begin{aligned} f(A) &\leq \frac{M1_H - A}{M-m} f(m) + \frac{A - m1_H}{M-m} f(M) - \delta_f \widetilde{A} \qquad (2.1) \\ \text{(resp.} \quad f(A) &\geq \frac{M1_H - A}{M-m} f(m) + \frac{A - m1_H}{M-m} f(M) + \delta_f \widetilde{A}\ ) \end{aligned}$$

holds for every continuous convex (resp. concave) function $f : [m, M] \to \mathbb{R}$, where

$$\delta_f = f(m) + f(M) - 2f\left(\tfrac{m+M}{2}\right) \quad \text{(resp. } \delta_f = 2f\left(\tfrac{m+M}{2}\right) - f(m) - f(M)\text{)},$$
$$\text{and} \quad \widetilde{A} = \tfrac{1}{2}1_H - \tfrac{1}{M-m}\left|A - \tfrac{m+M}{2}1_H\right|.$$

We shall give the proof for the convenience of the reader.

Proof of Lemma C. We prove only the convex case.

In [8, Theorem 1, p. 717] is prove that

$$\begin{array}{rl} & \min\{p_1,p_2\}\left[f(x)+f(y)-2f\left(\frac{x+y}{2}\right)\right] \\ \leq & p_1 f(x)+p_2 f(y)-f(p_1x+p_2y) \end{array} \tag{2.2}$$

holds for every convex function f on an interval I and $x,y\in I$, $p_1,p_2\in[0,1]$ such that $p_1+p_2=1$.

Putting $x=m, y=M$ in (2.2) it follows that

$$\begin{array}{rcl} f\left(p_1m+p_2M\right) & \leq & p_1f(m)+p_2f(M) \\ & - & \min\{p_1,p_2\}\left(f(m)+f(M)-2f\left(\frac{m+M}{2}\right)\right) \end{array} \tag{2.3}$$

holds for every $p_1,p_2\in[0,1]$ such that $p_1+p_2=1$. For any $t\in[m,M]$ we can write

$$f\left(t\right)=f\left(\frac{M-t}{M-m}m+\frac{t-m}{M-m}M\right).$$

Then by using (2.3) for $p_1=\frac{M-t}{M-m}$ and $p_2=\frac{t-m}{M-m}$ we get

$$\begin{array}{rcl} f(t) & \leq & \dfrac{M-t}{M-m}f(m)+\dfrac{t-m}{M-m}f(M) \\ & - & \left(\dfrac{1}{2}-\dfrac{1}{M-m}\left|t-\dfrac{m+M}{2}\right|\right)\left(f(m)+f(M)-2f\left(\dfrac{m+M}{2}\right)\right), \end{array} \tag{2.4}$$

since

$$\min\left\{\frac{M-t}{M-m},\frac{t-m}{M-m}\right\}=\frac{1}{2}-\frac{1}{M-m}\left|t-\frac{m+M}{2}\right|.$$

Finally we use the continuous functional calculus for a self-adjoint operator A: $f,g\in\mathcal{C}(I), Sp(A)\subseteq I$ and $f\geq g$ implies $f(A)\geq g(A)$; and $h(t)=|t|$ implies $h(A)=|A|$. Then by using (2.4) we obtain the desired inequality (2.1). □

In the following theorem we give an extension of Jensen's inequality given in Theorem B and a refinement of Theorem A.

Theorem 2.1. *Let $(A_1,\ldots,A_n)$ be an $n-$tuple of self-adjoint operators $A_i\in B(H)$ with the bounds m_i and M_i, $m_i\leq M_i$, $i=1,\ldots,n$. Let $(\Phi_1,\ldots,\Phi_n)$ be an $n-$tuple of positive linear mappings $\Phi_i: B(H)\to B(K)$, such that $\sum_{i=1}^{n_1}\Phi_i(1_H)=\alpha\,1_K$, $\sum_{i=n_1+1}^{n}\Phi_i(1_H)=\beta\,1_K$, where $1\leq n_1<n$, $\alpha,\beta>0$ and $\alpha+\beta=1$. Let $m_L=\min\{m_1,\ldots,m_{n_1}\}$, $M_R=\max\{M_1,\ldots,M_{n_1}\}$ and*

$$\begin{array}{rcl} m & = & \begin{cases} m_L, & \text{if } \{M_i\colon M_i\leq m_L, i\in\{n_1+1,\ldots,n\}\}=\emptyset, \\ \max\{M_i\colon M_i\leq m_L, i\in\{n_1+1,\ldots,n\}\}, & \text{otherwise,} \end{cases} \\ M & = & \begin{cases} M_R, & \text{if } \{m_i\colon m_i\geq M_R, i\in\{n_1+1,\ldots,n\}\}=\emptyset, \\ \min\{m_i\colon m_i\geq M_R, i\in\{n_1+1,\ldots,n\}\}, & \text{otherwise.} \end{cases} \end{array}$$

If

$$(m_L,M_R)\cap[m_i,M_i]=\emptyset \quad \textit{for} \quad i=n_1+1,\ldots,n, \qquad m<M,$$

and one of two equalities

$$\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(A_i)=\sum_{i=1}^{n}\Phi_i(A_i)=\frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(A_i)$$

is valid, then

$$\begin{aligned}\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f(A_i)) &\le \frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f(A_i))+\beta\delta_f\widetilde{A}\le\sum_{i=1}^{n}\Phi_i(f(A_i))\\ &\le \frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(f(A_i))-\alpha\delta_f\widetilde{A}\le\frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(f(A_i)),\end{aligned} \tag{2.5}$$

holds for every continuous convex function $f: I\to\mathbb{R}$ *provided that the interval* I *contains all* m_i, M_i, $i=1,\dots,n$, *where*

$$\begin{aligned}\delta_f\equiv\delta_f(\bar m,\bar M)&=f(\bar m)+f(\bar M)-2f\left(\frac{\bar m+\bar M}{2}\right)\\ \widetilde{A}\equiv\widetilde{A}_{A,\Phi,n_1,\alpha}(\bar m,\bar M)&=\frac{1}{2}1_K-\frac{1}{\alpha(\bar M-\bar m)}\sum_{i=1}^{n_1}\Phi_i\left(\left|A_i-\frac{\bar m+\bar M}{2}1_H\right|\right)\end{aligned} \tag{2.6}$$

and $\bar m\in[m,m_L]$, $\bar M\in[M_R,M]$, $\bar m<\bar M$, *are arbitrary numbers.*

If $f: I\to\mathbb{R}$ *is concave, then the reverse inequality is valid in* (2.5).

Proof. We prove only the convex case.

Let us denote

$$A=\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(A_i),\qquad B=\frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(A_i),\qquad C=\sum_{i=1}^{n}\Phi_i(A_i).$$

It is easy to verify that $A=B$ or $B=C$ or $A=C$ implies $A=B=C$.

Since f is convex on $[\bar m,\bar M]$ and $\mathsf{Sp}(A_i)\subseteq[m_i,M_i]\subseteq[\bar m,\bar M]$ for $i=1,\dots,n_1$, it follows from Lemma C that

$$f(A_i)\le\frac{\bar M1_H-A_i}{\bar M-\bar m}f(\bar m)+\frac{A_i-\bar m1_H}{\bar M-\bar m}f(\bar M)-\delta_f\widetilde{A}_i,\qquad i=1,\dots,n_1$$

holds, where $\delta_f=f(\bar m)+f(\bar M)-2f\left(\frac{\bar m+\bar M}{2}\right)$ and $\widetilde{A}_i=\frac{1}{2}1_H-\frac{1}{\bar M-\bar m}\left|A_i-\frac{\bar m+\bar M}{2}1_H\right|$. Applying a positive linear mapping Φ_i and summing, we obtain

$$\begin{aligned}\sum_{i=1}^{n_1}\Phi_i(f(A_i)) &\le \frac{\bar M\alpha1_K-\sum_{i=1}^{n_1}\Phi_i(A_i)}{\bar M-\bar m}f(\bar m)+\frac{\sum_{i=1}^{n_1}\Phi_i(A_i)-\bar m\alpha1_K}{\bar M-\bar m}f(\bar M)\\ &-\delta_f\left(\frac{\alpha}{2}1_K-\frac{1}{\bar M-\bar m}\sum_{i=1}^{n_1}\Phi_i\left(\left|A_i-\frac{\bar m+\bar M}{2}1_H\right|\right)\right),\end{aligned}$$

since $\sum_{i=1}^{n_1}\Phi_i(1_H)=\alpha1_K$. It follows that

$$\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f(A_i))\le\frac{\bar M1_K-A}{\bar M-\bar m}f(\bar m)+\frac{A-\bar m1_K}{\bar M-\bar m}f(\bar M)-\delta_f\widetilde{A}, \tag{2.7}$$

where $\widetilde{A}=\frac{1}{2}1_K-\frac{1}{\alpha(\bar M-\bar m)}\sum_{i=1}^{n_1}\Phi_i\left(\left|A_i-\frac{\bar m+\bar M}{2}1_H\right|\right)$.

In addition, since f is convex on all $[m_i,M_i]$ and $(\bar m,\bar M)\cap[m_i,M_i]=\emptyset$ for $i=n_1+1,\dots,n$, then

$$f(A_i)\ge\frac{\bar M1_H-A_i}{\bar M-\bar m}f(\bar m)+\frac{A_i-\bar m1_H}{\bar M-\bar m}f(\bar M),\qquad i=n_1+1,\dots,n.$$

It follows

$$\frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i\left(f(A_i)\right)-\delta_f\widetilde{A}\geq\frac{\bar{M}1_K-B}{\bar{M}-\bar{m}}f(\bar{m})+\frac{B-\bar{m}1_K}{\bar{M}-\bar{m}}f(\bar{M})-\delta_f\widetilde{A}. \tag{2.8}$$

Combining (2.7) and (2.8) and taking into account that $A=B$, we obtain

$$\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i\left(f(A_i)\right)\leq\frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i\left(f(A_i)\right)-\delta_f\widetilde{A}. \tag{2.9}$$

Next, we obtain

$$\begin{aligned}
&\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f(A_i))\\
=\ &\sum_{i=1}^{n_1}\Phi_i(f(A_i))+\frac{\beta}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f(A_i)) \quad (\text{by } \alpha+\beta=1)\\
\leq\ &\sum_{i=1}^{n_1}\Phi_i(f(A_i))+\sum_{i=n_1+1}^{n}\Phi_i(f(A_i))-\beta\delta_f\widetilde{A} \qquad\qquad (\text{by } (2.9))\\
\leq\ &\frac{\alpha}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(f(A_i))-\alpha\delta_f\widetilde{A}+\sum_{i=n_1+1}^{n}\Phi_i(f(A_i))-\beta\delta_f\widetilde{A} \qquad (\text{by } (2.9))\\
=\ &\frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(f(A_i))-\delta_f\widetilde{A} \quad (\text{by } \alpha+\beta=1),
\end{aligned}$$

which gives the following double inequality

$$\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f(A_i))\leq\sum_{i=1}^{n}\Phi_i(f(A_i))-\beta\delta_f\widetilde{A}\leq\frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(f(A_i))-\delta_f\widetilde{A}.$$

Adding $\beta\delta_f\widetilde{A}$ in the above inequalities, we get

$$\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f(A_i))+\beta\delta_f\widetilde{A}\leq\sum_{i=1}^{n}\Phi_i(f(A_i))\leq\frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(f(A_i))-\alpha\delta_f\widetilde{A}. \tag{2.10}$$

Now, we remark that $\delta_f\geq 0$ and $\widetilde{A}\geq 0$. (Indeed, since f is convex, then $f\left((\bar{m}+\bar{M})/2\right)\leq(f(\bar{m})+f(\bar{M}))/2$, which implies that $\delta_f\geq 0$. Also, since

$$\mathsf{Sp}(A_i)\subseteq[\bar{m},\bar{M}] \quad\Rightarrow\quad \left|A_i-\frac{\bar{M}+\bar{m}}{2}1_H\right|\leq\frac{\bar{M}-\bar{m}}{2}1_H, \qquad \text{for } i=1,\dots,n_1,$$

then

$$\sum_{i=1}^{n_1}\Phi_i\left(\left|A_i-\frac{\bar{M}+\bar{m}}{2}1_H\right|\right)\leq\frac{\bar{M}-\bar{m}}{2}\alpha 1_K,$$

which gives

$$0\leq\frac{1}{2}1_K-\frac{1}{\alpha(\bar{M}-\bar{m})}\sum_{i=1}^{n_1}\Phi_i\left(\left|A_i-\frac{\bar{M}+\bar{m}}{2}1_H\right|\right)=\widetilde{A}. \qquad)$$

Consequently, the following inequalities

$$\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f(A_i)) \le \frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f(A_i)) + \beta\delta_f\widetilde{A},$$
$$\frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(f(A_i)) - \alpha\delta_f\widetilde{A} \le \frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(f(A_i)),$$

hold, which with (2.10) proves the desired series inequalities (2.5).

□

Example 2.2. We observe the matrix case of Theorem 2.1 for $f(t) = t^4$, which is the convex function but not operator convex, $n = 4$, $n_1 = 2$ and the bounds of matrices as in Figure 1.

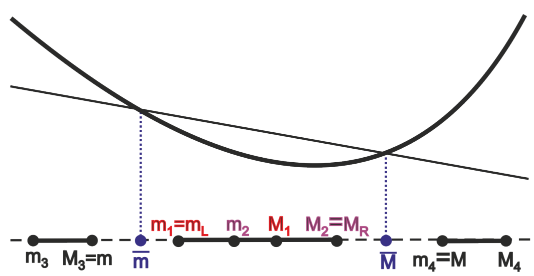

FIGURE 1. An example a convex function and the bounds of four operators

We show an example such that

$$\begin{aligned}\tfrac{1}{\alpha}\left(\Phi_1(A_1^4)+\Phi_2(A_2^4)\right) &< \tfrac{1}{\alpha}\left(\Phi_1(A_1^4)+\Phi_2(A_2^4)\right)+\beta\delta_f\widetilde{A}\\ &< \Phi_1(A_1^4)+\Phi_2(A_2^4)+\Phi_3(A_3^4)+\Phi_4(A_4^4)\\ &< \tfrac{1}{\beta}\left(\Phi_3(A_3^4)+\Phi_4(A_4^4)\right)-\alpha\delta_f\widetilde{A} < \tfrac{1}{\beta}\left(\Phi_3(A_3^4)+\Phi_4(A_4^4)\right)\end{aligned} \tag{2.11}$$

holds, where $\delta_f = \bar{M}^4 + \bar{m}^4 - (\bar{M}+\bar{m})^4/8$ and

$$\widetilde{A} = \frac{1}{2}I_2 - \frac{1}{\alpha(\bar{M}-\bar{m})}\left(\Phi_1\left(|A_1 - \frac{\bar{M}+\bar{m}}{2}I_h|\right) + \Phi_2\left(|A_2 - \frac{\bar{M}+\bar{m}}{2}I_3|\right)\right).$$

We define mappings $\Phi_i : M_3(\mathbb{C}) \to M_2(\mathbb{C})$ as follows: $\Phi_i((a_{jk})_{1\le j,k\le 3}) = \frac{1}{4}(a_{jk})_{1\le j,k\le 2}$, $i = 1,\dots,4$. Then $\sum_{i=1}^4 \Phi_i(I_3) = I_2$ and $\alpha = \beta = \frac{1}{2}$.
Let

$$A_1 = 2\begin{pmatrix}2 & 9/8 & 1\\ 9/8 & 2 & 0\\ 1 & 0 & 3\end{pmatrix}, \qquad A_2 = 3\begin{pmatrix}2 & 9/8 & 0\\ 9/8 & 1 & 0\\ 0 & 0 & 2\end{pmatrix},$$
$$A_3 = -3\begin{pmatrix}4 & 1/2 & 1\\ 1/2 & 4 & 0\\ 1 & 0 & 2\end{pmatrix}, \qquad A_4 = 12\begin{pmatrix}5/3 & 1/2 & 0\\ 1/2 & 3/2 & 0\\ 0 & 0 & 3\end{pmatrix}.$$

Then $m_1 = 1.28607$, $M_1 = 7.70771$, $m_2 = 0.53777$, $M_2 = 5.46221$, $m_3 = -14.15050$, $M_3 = -4.71071$, $m_4 = 12.91724$, $M_4 = 36.$, so $m_L = m_2$, $M_R = M_1$,

$m = M_3$ and $M = m_4$ (rounded to five decimal places). Also,

$$\frac{1}{\alpha}\left(\Phi_1(A_1)+\Phi_2(A_2)\right) = \frac{1}{\beta}\left(\Phi_3(A_3)+\Phi_4(A_4)\right) = \begin{pmatrix} 4 & 9/4 \\ 9/4 & 3 \end{pmatrix},$$

and

$$\begin{aligned} A_f \equiv \frac{1}{\alpha}\left(\Phi_1(A_1^4)+\Phi_2(A_2^4)\right) &= \begin{pmatrix} 989.00391 & 663.46875 \\ 663.46875 & 526.12891 \end{pmatrix}, \\ C_f \equiv \Phi_1(A_1^4)+\Phi_2(A_2^4)+\Phi_3(A_3^4)+\Phi_4(A_4^4) &= \begin{pmatrix} 68093.14258 & 48477.98437 \\ 48477.98437 & 51335.39258 \end{pmatrix}, \\ B_f \equiv \frac{1}{\beta}\left(\Phi_3(A_3^4)+\Phi_4(A_4^4)\right) &= \begin{pmatrix} 135197.28125 & 96292.5 \\ 96292.5 & 102144.65625 \end{pmatrix}. \end{aligned}$$

Then

$$A_f < C_f < B_f \tag{2.12}$$

holds (which is consistent with (1.3)).

We will choose three pairs of numbers $(\bar{m}, \bar{M})$, $\bar{m} \in [-4.71071, 0.53777]$, $\bar{M} \in [7.70771, 12.91724]$ as follows:

i) $\bar{m} = m_L = 0.53777$, $\bar{M} = M_R = 7.70771$, then

$$\widetilde{\Delta}_1 = \beta\delta_f\widetilde{A} = 0.5\cdot 2951.69249\cdot\begin{pmatrix} 0.15678 & 0.09030 \\ 0.09030 & 0.15943 \end{pmatrix} = \begin{pmatrix} 231.38908 & 133.26139 \\ 133.26139 & 235.29515 \end{pmatrix},$$

ii) $\bar{m} = m = -4.71071$, $\bar{M} = M = 12.91724$, then

$$\widetilde{\Delta}_2 = \beta\delta_f\widetilde{A} = 0.5\cdot 27766.07963\cdot\begin{pmatrix} 0.36022 & 0.03573 \\ 0.03573 & 0.36155 \end{pmatrix} = \begin{pmatrix} 5000.89860 & 496.04498 \\ 496.04498 & 5019.50711 \end{pmatrix},$$

iii) $\bar{m} = -1$, $\bar{M} = 10$, then

$$\widetilde{\Delta}_3 = \beta\delta_f\widetilde{A} = 0.5\cdot 9180.875\cdot\begin{pmatrix} 0.28203 & 0.08975 \\ 0.08975 & 0.27557 \end{pmatrix} = \begin{pmatrix} 1294.66 & 411.999 \\ 411.999 & 1265. \end{pmatrix}.$$

New, we obtain the following improvement of (2.12) (see (2.11)):

i)
$$\begin{aligned} A_f < A_f + \widetilde{\Delta}_1 &= \begin{pmatrix} 1220.39299 & 796.73014 \\ 796.73014 & 761.42406 \end{pmatrix} \\ < C_f < \begin{pmatrix} 134965.89217 & 96159.23861 \\ 96159.23861 & 101909.36110 \end{pmatrix} &= B_f - \widetilde{\Delta}_1 < B_f, \end{aligned}$$

ii)
$$\begin{aligned} A_f < A_f + \widetilde{\Delta}_2 &= \begin{pmatrix} 5989.90251 & 1159.51373 \\ 1159.51373 & 5545.63601 \end{pmatrix} \\ < C_f < \begin{pmatrix} 130196.38265 & 95796.45502 \\ 95796.45502 & 97125.14914 \end{pmatrix} &= B_f - \widetilde{\Delta}_2 < B_f, \end{aligned}$$

iii)
$$\begin{aligned} A_f < A_f + \widetilde{\Delta}_3 &= \begin{pmatrix} 2283.66362 & 1075.46746 \\ 1075.46746 & 1791.12874 \end{pmatrix} \\ < C_f < \begin{pmatrix} 133902.62153 & 95880.50129 \\ 95880.50129 & 100879.65641 \end{pmatrix} &= B_f - \widetilde{\Delta}_3 < B_f. \end{aligned}$$

Using Theorem 2.1 we get the following result.

Corollary 2.3. *Let the assumptions of Theorem 2.1 hold. Then*

$$\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f(A_i)) \le \frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f(A_i)) + \gamma_1\delta_f\widetilde{A} \le \frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(f(A_i)) \tag{2.13}$$

and

$$\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f(A_i)) \le \frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(f(A_i)) - \gamma_2\delta_f\widetilde{A} \le \frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(f(A_i)) \tag{2.14}$$

holds for every γ_1, γ_2 in the close interval joining α and β, where δ_f and $\widetilde{A}$ are defined by (2.6).

Proof. Adding $\alpha\delta_f\widetilde{A}$ in (2.5) and noticing $\delta_f\widetilde{A} \ge 0$, we obtain

$$\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f(A_i)) \le \frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f(A_i)) + \alpha\delta_f\widetilde{A} \le \frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(f(A_i)).$$

Taking into account the above inequality and the left hand side of (2.5) we obtain (2.13).

Similarly, subtracting $\beta\delta_f\widetilde{A}$ in (2.5) we obtain (2.14). □

Remark 2.4. *Let the assumptions of Theorem 2.1 be valid.*

1) We observe that the following inequality

$$f\left(\frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(A_i)\right) \le \frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(f(A_i)) - \delta_f\widetilde{A}_\beta \le \frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(f(A_i)),$$

holds for every continuous convex function $f : I \to \mathbb{R}$ provided that the interval I contains all m_i, M_i, $i = 1, \ldots, n$, where δ_f is defined by (2.6),

$$\widetilde{A}_\beta \equiv \widetilde{A}_{\beta,A,\Phi,n_1}(\bar{m}, \bar{M}) = \frac{1}{2}1_K - \frac{1}{\bar{M}-\bar{m}}\left|\frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i A_i - \frac{\bar{m}+\bar{M}}{2}1_K\right|$$

and $\bar{m} \in [m, m_L]$, $\bar{M} \in [M_R, M]$, $\bar{m} < \bar{M}$, are arbitrary numbers.

Indeed, by the assumptions of Theorem 2.1 we have

$$m_L\alpha 1_H \le \sum_{i=1}^{n_1}\Phi_i(A_i) \le M_R\alpha 1_H \quad \text{and} \quad \frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(A_i) = \frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(A_i)$$

which implies

$$m_L 1_H \le \frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(A_i) \le M_R 1_H.$$

Also $(m_L, M_R) \cap [m_i, M_i] = \emptyset$ for $i = n_1+1, \ldots, n$ and $\sum_{i=n_1+1}^{n}\frac{1}{\beta}\Phi_i(1_H) = 1_K$ hold. So we can apply Theorem B on operators $A_{n_1+1}, \ldots, A_n$ and mappings $\frac{1}{\beta}\Phi_i$. We obtain the desired inequality.

2) We denote by m_C and M_C the bounds of $C=\sum_{i=1}^{n}\Phi_i(A_i)$. If $(m_C,M_C)\cap[m_i,M_i]=\emptyset$, $i=1,\dots,n_1$, then series inequality (2.5) can be extended from the left side if we use refined Jensen's operator inequality (1.4)

$$\begin{aligned} f\left(\sum_{i=1}^{n}\Phi_i(A_i)\right) &= f\left(\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(A_i)\right) \le \frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f(A_i)) - \delta_f\widetilde{A}_\alpha \\ \le & \frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f(A_i)) \le \frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f(A_i)) + \beta\delta_f\widetilde{A} \le \sum_{i=1}^{n}\Phi_i(f(A_i)) \\ \le & \frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(f(A_i)) - \alpha\delta_f\widetilde{A} \le \frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(f(A_i)), \end{aligned}$$

where δ_f and $\widetilde{A}$ are defined by (2.6),

$$\widetilde{A}_\alpha \equiv \widetilde{A}_{\alpha,A,\Phi,n_1}(\bar{m},\bar{M}) = \frac{1}{2}1_K - \frac{1}{\bar{M}-\bar{m}}\left|\frac{1}{\alpha}\sum_{i=n_1+1}^{n}\Phi_iA_i - \frac{\bar{m}+\bar{M}}{2}1_K\right|$$

Remark 2.5. We obtain the equivalent inequalities to the ones in Theorem 2.1 in the case when $\sum_{i=1}^{n}\Phi_i(1_H)=\gamma\,1_K$, for some positive scalar γ. If $\alpha+\beta=\gamma$ and one of two equalities

$$\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(A_i) = \frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(A_i) = \frac{1}{\gamma}\sum_{i=1}^{n}\Phi_i(A_i)$$

is valid, then

$$\begin{aligned} \frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f(A_i)) &\le \frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f(A_i)) + \frac{\beta}{\gamma}\delta_f\widetilde{A} \le \frac{1}{\gamma}\sum_{i=1}^{n}\Phi_i(f(A_i)) \\ &\le \frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(f(A_i)) - \frac{\alpha}{\gamma}\delta_f\widetilde{A} \le \frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(f(A_i)), \end{aligned}$$

holds for every continuous convex function $f: I\to\mathbb{R}$ provided that the interval I contains all m_i, M_i, $i=1,\dots,n$, where δ_f and $\widetilde{A}$ are defined by (2.6).

With respect to Remark 2.5, we obtain the following obvious corollary of Theorem 2.1 with the convex combination of operators A_i, $i=1,\dots,n$.

Corollary 2.6. *Let* $(A_1,\dots,A_n)$ *be an* $n-$*tuple of self-adjoint operators* $A_i\in B(H)$ *with the bounds* m_i *and* M_i, $m_i\le M_i$, $i=1,\dots,n$. *Let* $(p_1,\dots,p_n)$ *be an* $n-$*tuple of non-negative numbers such that* $0<\sum_{i=1}^{n_1}p_i=\mathbf{p_{n_1}}<\mathbf{p_n}=\sum_{i=1}^{n}p_i$, *where* $1\le n_1<n$. *Let*
$m_L=\min\{m_1,\dots,m_{n_1}\}$, $M_R=\max\{M_1,\dots,M_{n_1}\}$ *and*

$$\begin{aligned} m &= \begin{cases} m_L, & \text{if } \{M_i\colon M_i\le m_L, i\in\{n_1+1,\dots,n\}\}=\emptyset, \\ \max\{M_i\colon M_i\le m_L, i\in\{n_1+1,\dots,n\}\}, & \text{otherwise,} \end{cases} \\ M &= \begin{cases} M_R, & \text{if } \{m_i\colon m_i\ge M_R, i\in\{n_1+1,\dots,n\}\}=\emptyset, \\ \min\{m_i\colon m_i\ge M_R, i\in\{n_1+1,\dots,n\}\}, & \text{otherwise.} \end{cases} \end{aligned}$$

If

$$(m_L, M_R) \cap [m_i, M_i] = \emptyset \quad \textit{for} \quad i = n_1+1, \dots, n, \qquad m < M,$$

and one of two equalities

$$\frac{1}{\mathbf{p_{n_1}}} \sum_{i=1}^{n_1} p_i A_i = \frac{1}{\mathbf{p_n}} \sum_{i=1}^{n} p_i A_i = \frac{1}{\mathbf{p_n} - \mathbf{p_{n_1}}} \sum_{i=n_1+1}^{n} p_i A_i$$

is valid, then

$$\begin{aligned} \frac{1}{\mathbf{p_{n_1}}} \sum_{i=1}^{n_1} p_i f(A_i) &\leq \frac{1}{\mathbf{p_{n_1}}} \sum_{i=1}^{n_1} p_i f(A_i) + \left(1 - \frac{\mathbf{p_{n_1}}}{\mathbf{p_n}}\right) \delta_f \widetilde{A} \leq \frac{1}{\mathbf{p_n}} \sum_{i=1}^{n} p_i f(A_i) \\ &\leq \frac{1}{\mathbf{p_n} - \mathbf{p_{n_1}}} \sum_{i=n_1+1}^{n} p_i f(A_i) - \frac{\mathbf{p_{n_1}}}{\mathbf{p_n}} \delta_f \widetilde{A} \leq \frac{1}{\mathbf{p_n} - \mathbf{p_{n_1}}} \sum_{i=n_1+1}^{n} p_i f(A_i), \end{aligned} \tag{2.15}$$

holds for every continuous convex function $f : I \to \mathbb{R}$ provided that the interval I contains all m_i, M_i, $i = 1, \dots, n$, where where δ_f is defined by (2.6),

$$\widetilde{A} \equiv \widetilde{A}_{A,p,n_1}(\bar{m}, \bar{M}) = \frac{1}{2} 1_H - \frac{1}{\mathbf{p_{n_1}}(\bar{M} - \bar{m})} \sum_{i=1}^{n_1} p_i \left(\left| A_i - \frac{\bar{m} + \bar{M}}{2} 1_H \right| \right)$$

and $\bar{m} \in [m, m_L]$, $\bar{M} \in [M_R, M]$, $\bar{m} < \bar{M}$, *are arbitrary numbers.*

If $f : I \to \mathbb{R}$ is concave, then the reverse inequality is valid in (2.15).

As a special case of Corollary 2.6 we obtain an extension of [7, Corollary 6].

Corollary 2.7. *Let $(A_1, \dots, A_n)$ be an $n-$tuple of self-adjoint operators $A_i \in B(H)$ with the bounds m_i and M_i, $m_i \leq M_i$, $i = 1, \dots, n$. Let $(p_1, \dots, p_n)$ be an $n-$tuple of non-negative numbers such that $\sum_{i=1}^n p_i = 1$. Let*

$$(m_A, M_A) \cap [m_i, M_i] = \emptyset \quad \textit{for } i = 1, \dots, n, \qquad \textit{and} \qquad m < M,$$

where m_A and M_A, $m_A \leq M_A$, are the bounds of $A = \sum_{i=1}^n p_i A_i$ and

$$m = \max\left\{M_i \leq m_A, i \in \{1, \dots, n\}\right\}, \ M = \min\left\{m_i \geq M_A, i \in \{1, \dots, n\}\right\}.$$

If $f : I \to \mathbb{R}$ is a continuous convex function provided that the interval I contains all m_i, M_i, then

$$\begin{aligned} f\Big(\sum_{i=1}^n p_i A_i\Big) &\leq f\Big(\sum_{i=1}^n p_i A_i\Big) + \frac{1}{2} \delta_f \tilde{\tilde{A}} \leq \frac{1}{2} f\Big(\sum_{i=1}^n p_i A_i\Big) + \frac{1}{2} \sum_{i=1}^n p_i f(A_i) \\ &\leq \sum_{i=1}^n p_i f(A_i) - \frac{1}{2} \delta_f \tilde{\tilde{A}} \leq \sum_{i=1}^n p_i f(A_i), \end{aligned} \tag{2.16}$$

holds, where δ_f is defined by (2.6), $\tilde{\tilde{A}} = \frac{1}{2} 1_H - \frac{1}{\bar{M} - \bar{m}} \left| \sum_{i=1}^n p_i A_i - \frac{\bar{m} + \bar{M}}{2} 1_H \right|$ and $\bar{m} \in [m, m_A]$, $\bar{M} \in [M_A, M]$, $\bar{m} < \bar{M}$, are arbitrary numbers.

If $f : I \to \mathbb{R}$ is concave, then the reverse inequality is valid in (2.16).

Proof. We prove only the convex case.

We define $(n+1)-$tuple of operators $(B_1,\ldots,B_{n+1})$, $B_i \in B(H)$, by $B_1 = A = \sum_{i=1}^n p_i A_i$ and $B_i = A_{i-1}$, $i=2,\ldots,n+1$. Then $m_{B_1} = m_A$, $M_{B_1} = M_A$ are the bounds of B_1 and $m_{B_i} = m_{i-1}$, $M_{B_i} = M_{i-1}$ are the ones of B_i, $i=2,\ldots,n+1$. Also, we define $(n+1)-$tuple of non-negative numbers $(q_1,\ldots,q_{n+1})$ by $q_1 = 1$ and $q_i = p_{i-1}$, $i=2,\ldots,n+1$. We have that $\sum_{i=1}^{n+1} q_i = 2$ and

$$(m_{B_1}, M_{B_1}) \cap [m_{B_i}, M_{B_i}] = \emptyset, \quad \text{for } i=2,\ldots,n+1 \quad \text{and} \quad m<M \tag{2.17}$$

holds. Since

$$\sum_{i=1}^{n+1} q_i B_i = B_1 + \sum_{i=2}^{n+1} q_i B_i = \sum_{i=1}^{n} p_i A_i + \sum_{i=1}^{n} p_i A_i = 2B_1,$$

then

$$q_1 B_1 = \frac{1}{2}\sum_{i=1}^{n+1} q_i B_i = \sum_{i=2}^{n+1} q_i B_i. \tag{2.18}$$

Taking into account (2.17) and (2.18), we can apply Corollary 2.6 for $n_1 = 1$ and B_i, q_i as above, and we get

$$q_1 f(B_1) \le q_1 f(B_1) + \frac{1}{2}\delta_f \widetilde{B} \le \frac{1}{2}\sum_{i=1}^{n+1} q_i f(B_i) \le \sum_{i=2}^{n+1} q_i f(B_i) - \frac{1}{2}\delta_f \widetilde{B} \le \sum_{i=2}^{n+1} q_i f(B_i),$$

where $\widetilde{B} = \frac{1}{2}1_H - \frac{1}{\bar{M}-\bar{m}}\left|B_1 - \frac{\bar{m}+\bar{M}}{2}1_H\right|$, which gives the desired inequality (2.16). □

3. Quasi-arithmetic means

In this section we study an application of Theorem 2.1 to the quasi-arithmetic mean with weight.

For a subset $\{A_{n_1},\ldots,A_{n_2}\}$ of $\{A_1,\ldots,A_n\}$, we denote the quasi-arithmetic mean by

$$\mathcal{M}_\varphi(\gamma, \mathbf{A}, \mathbf{\Phi}, n_1, n_2) = \varphi^{-1}\left(\frac{1}{\gamma}\sum_{i=n_1}^{n_2} \Phi_i\left(\varphi(A_i)\right)\right), \tag{3.1}$$

where $(A_{n_1},\ldots,A_{n_2})$ are self-adjoint operators in $\mathcal{B}(H)$ with the spectra in I, $(\Phi_{n_1},\ldots,\Phi_{n_2})$ are positive linear mappings $\Phi_i : \mathcal{B}(H) \to \mathcal{B}(K)$ such that $\sum_{i=n_1}^{n_2} \Phi_i(1_H) = \gamma\, 1_K$, and $\varphi : I \to \mathbb{R}$ is a continuous strictly monotone function.

Under the same conditions, for convenience we introduce the following denotations

$$\begin{aligned} \delta_{\varphi,\psi}(m,M) &= \psi(m)+\psi(M) - 2\psi\circ\varphi^{-1}\left(\tfrac{\varphi(m)+\varphi(M)}{2}\right), \\ \widetilde{A}_{\varphi,n_1,\gamma}(m,M) &= \tfrac{1}{2}1_K - \tfrac{1}{\gamma(M-m)}\textstyle\sum_{i=1}^{n_1} \Phi_i\left(\left|\varphi(A_i) - \tfrac{\varphi(M)+\varphi(m)}{2}1_H\right|\right), \end{aligned} \tag{3.2}$$

where $\varphi,\psi : I \to \mathbb{R}$ are continuous strictly monotone functions and $m, M \in I$, $m<M$. Of course, we include implicitly that $\widetilde{A}_{\varphi,n_1,\gamma}(m,M) \equiv \widetilde{A}_{\varphi,A,\Phi,n_1,\gamma}(m,M)$.

The following theorem is an extension of [7, Theorem 7] and a refinement of [6, Theorem 3.1].

Theorem 3.1. *Let $(A_1,\dots,A_n)$ be an $n-$tuple of self-adjoint operators $A_i \in B(H)$ with the bounds m_i and M_i, $m_i \le M_i$, $i=1,\dots,n$. Let $\varphi,\psi : I \to \mathbb{R}$ be continuous strictly monotone functions on an interval I which contains all m_i, M_i. Let $(\Phi_1,\dots,\Phi_n)$ be an $n-$tuple of positive linear mappings $\Phi_i : B(H) \to B(K)$, such that $\sum_{i=1}^{n_1}\Phi_i(1_H) = \alpha\, 1_K$, $\sum_{i=n_1+1}^{n}\Phi_i(1_H) = \beta\, 1_K$, where $1 \le n_1 < n$, $\alpha,\beta>0$ and $\alpha+\beta=1$. Let one of two equalities*

$$\mathcal{M}_\varphi(\alpha,\mathbf{A},\mathbf{\Phi},1,n_1) = \mathcal{M}_\varphi(1,\mathbf{A},\mathbf{\Phi},1,n) = \mathcal{M}_\varphi(\beta,\mathbf{A},\mathbf{\Phi},n_1+1,n) \tag{3.3}$$

be valid and let

$$(m_L, M_R)\cap[m_i,M_i]=\emptyset \quad \text{for} \quad i=n_1+1,\dots,n, \qquad m<M,$$

where $m_L=\min\{m_1,\dots,m_{n_1}\}$, $M_R=\max\{M_1,\dots,M_{n_1}\}$,

$$\begin{aligned} m &= \begin{cases} m_L, & \text{if } \{M_i\colon M_i \le m_L, i\in\{n_1+1,\dots,n\}\}=\emptyset,\\ \max\{M_i\colon M_i\le m_L, i\in\{n_1+1,\dots,n\}\}, & \text{otherwise,}\end{cases}\\ M &= \begin{cases} M_R, & \text{if } \{m_i\colon m_i \ge M_R, i\in\{n_1+1,\dots,n\}\}=\emptyset,\\ \min\{m_i\colon m_i\ge M_R, i\in\{n_1+1,\dots,n\}\}, & \text{otherwise.}\end{cases}\end{aligned}$$

(i) *If $\psi\circ\varphi^{-1}$ is convex and ψ^{-1} is operator monotone, then*

$$\begin{aligned}\mathcal{M}_\psi(\alpha,\mathbf{A},\mathbf{\Phi},1,n_1) &\le \psi^{-1}\left(\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i\left(\psi(A_i)\right)+\beta\delta_{\varphi,\psi}\widetilde{A}_{\varphi,n_1,\alpha}\right) \le \mathcal{M}_\psi(1,\mathbf{A},\mathbf{\Phi},1,n)\\ &\le \psi^{-1}\left(\frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i\left(\psi(A_i)\right)-\alpha\delta_{\varphi,\psi}\widetilde{A}_{\varphi,n_1,\alpha}\right) \le \mathcal{M}_\psi(\beta,\mathbf{A},\mathbf{\Phi},n_1+1,n)\end{aligned} \tag{3.4}$$

holds, where $\delta_{\varphi,\psi}\ge 0$ and $\widetilde{A}_{\varphi,n_1,\alpha}\ge 0$.

(i′) *If $\psi\circ\varphi^{-1}$ is convex and $-\psi^{-1}$ is operator monotone, then the reverse inequality is valid in (3.4), where $\delta_{\varphi,\psi}\ge 0$ and $\widetilde{A}_{\varphi,n_1,\alpha}\ge 0$.*

(ii) *If $\psi\circ\varphi^{-1}$ is concave and $-\psi^{-1}$ is operator monotone, then (3.4) holds, where $\delta_{\varphi,\psi}\le 0$ and $\widetilde{A}_{\varphi,n_1,\alpha}\ge 0$.*

(ii′) *If $\psi\circ\varphi^{-1}$ is concave and ψ^{-1} is operator monotone, then the reverse inequality is valid in (3.4), where $\delta_{\varphi,\psi}\le 0$ and $\widetilde{A}_{\varphi,n_1,\alpha}\ge 0$.*

In all the above cases, we assume that $\delta_{\varphi,\psi}\equiv\delta_{\varphi,\psi}(\bar{m},\bar{M})$, $\widetilde{A}_{\varphi,n_1,\alpha}\equiv\widetilde{A}_{\varphi,n_1,\alpha}(\bar{m},\bar{M})$ are defined by (3.2) and $\bar{m}\in[m,m_L]$, $\bar{M}\in[M_R,M]$, $\bar{m}<\bar{M}$, are arbitrary numbers.

Proof. We only prove the case (i). Suppose that φ is a strictly increasing function. Then

$$(m_L,M_R)\cap[m_i,M_i]=\emptyset \qquad \text{for} \quad i=n_1+1,\dots,n$$

implies

$$(\varphi(m_L),\varphi(M_R))\cap[\varphi(m_i),\varphi(M_i)]=\emptyset \qquad \text{for } i=n_1+1,\dots,n. \tag{3.5}$$

Also, by using (3.3), we have

$$\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i\left(\varphi(A_i)\right)=\sum_{i=1}^{n}\Phi_i\left(\varphi(A_i)\right)=\frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i\left(\varphi(A_i)\right).$$

Taking into account (3.5) and the above double equality, we obtain by Theorem 2.1

$$\begin{aligned}\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f\left(\varphi(A_i)\right))&\le\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f\left(\varphi(A_i)\right))+\beta\delta_f\widetilde{A}_{\varphi,n_1,\alpha}\le\sum_{i=1}^{n}\Phi_i(f\left(\varphi(A_i)\right))\\&\le\frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(f\left(\varphi(A_i)\right))-\alpha\delta_f\widetilde{A}_{\varphi,n_1,\alpha}\le\frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(f\left(\varphi(A_i)\right)),\end{aligned}\tag{3.6}$$

for every continuous convex function $f:J\to\mathbb{R}$ on an interval J which contains all $[\varphi(m_i),\varphi(M_i)]=\varphi([m_i,M_i])$, $i=1,\dots,n$, where $\delta_f=f(\varphi(m))+f(\varphi(M))-2f\left(\frac{\varphi(m)+\varphi(M)}{2}\right)$.

Also, if φ is strictly decreasing, then we check that (3.6) holds for convex $f:J\to\mathbb{R}$ on J which contains all $[\varphi(M_i),\varphi(m_i)]=\varphi([m_i,M_i])$.

Putting $f=\psi\circ\varphi^{-1}$ in (3.6), we obtain

$$\begin{aligned}\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i\left(\psi(A_i)\right)&\le\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i\left(\psi(A_i)\right)+\beta\delta_{\varphi,\psi}\widetilde{A}_{\varphi,n_1,\alpha}\le\sum_{i=1}^{n}\Phi_i\left(\psi(A_i)\right)\\&\le\frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i\left(\psi(A_i)\right)-\alpha\delta_{\varphi,\psi}\widetilde{A}_{\varphi,n_1,\alpha}\le\frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i\left(\psi(A_i)\right).\end{aligned}$$

Applying an operator monotone function ψ^{-1} on the above double inequality, we obtain the desired inequality (3.4). □

We now give some particular results of interest that can be derived from Theorem 3.1, which are an extension of [7, Corollary 8, Corollary 10] and a refinement of [6, Corollary 3.3].

Corollary 3.2. *Let* $(A_1,\dots,A_n)$ *and* $(\Phi_1,\dots,\Phi_n)$, m_i, M_i, m, M, m_L, M_R, α *and* β *be as in Theorem 3.1. Let* I *be an interval which contains all* m_i,M_i *and*

$$(m_L,M_R)\cap[m_i,M_i]=\emptyset\quad\textit{for}\quad i=n_1+1,\dots,n,\qquad m<M.$$

I) *If one of two equalities*

$$\mathcal{M}_\varphi(\alpha,\mathbf{A},\mathbf{\Phi},1,n_1)=\mathcal{M}_\varphi(1,\mathbf{A},\mathbf{\Phi},1,n)=\mathcal{M}_\varphi(\beta,\mathbf{A},\mathbf{\Phi},n_1+1,n)$$

is valid, then

$$\begin{aligned}\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(A_i)&\le\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(A_i)+\beta\delta_{\varphi^{-1}}\widetilde{A}_{\varphi,n_1,\alpha}\le\sum_{i=1}^{n}\Phi_i(A_i)\\&\le\frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(A_i)-\alpha\delta_{\varphi^{-1}}\widetilde{A}_{\varphi,n_1,\alpha}\le\frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i\Phi_i(A_i).\end{aligned}\tag{3.7}$$

holds for every continuous strictly monotone function $\varphi : I \to \mathbb{R}$ *such that* φ^{-1} *is convex on* I*, where* $\delta_{\varphi^{-1}} = \bar{m} + \bar{M} - 2\,\varphi^{-1}\left(\frac{\varphi(\bar{m})+\varphi(\bar{M})}{2}\right) \geq 0$, $\widetilde{A}_{\varphi,n_1,\alpha} = \frac{1}{2}1_K - \frac{1}{\alpha(\bar{M}-\bar{m})}\sum_{i=1}^{n_1}\Phi_i\left(\left|\varphi(A_i) - \frac{\varphi(\bar{M})+\varphi(\bar{m})}{2}1_H\right|\right)$ *and* $\bar{m} \in [m, m_L]$, $\bar{M} \in [M_R, M]$, $\bar{m} < \bar{M}$*, are arbitrary numbers.*

But, if φ^{-1} *is concave, then the reverse inequality is valid in* (3.7) *for* $\delta_{\varphi^{-1}} \leq 0$.

II) *If one of two equalities*

$$\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(A_i) = \sum_{i=1}^{n}\Phi_i(A_i) = \frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(A_i)$$

is valid, then

$$\begin{aligned}\mathcal{M}_\varphi(\alpha, \mathbf{A}, \mathbf{\Phi}, 1, n_1) &\leq \varphi^{-1}\left(\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i\left(\varphi(A_i)\right) + \beta\delta_\varphi\widetilde{A}_{n_1}\right) \leq \mathcal{M}_\varphi(1, \mathbf{A}, \mathbf{\Phi}, 1, n)\\ &\leq \varphi^{-1}\left(\frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i\left(\varphi(A_i)\right) - \alpha\delta_\varphi\widetilde{A}_{n_1}\right) \leq \mathcal{M}_\varphi(\beta, \mathbf{A}, \mathbf{\Phi}, n_1+1, n)\end{aligned} \tag{3.8}$$

holds for every continuous strictly monotone function $\varphi : I \to \mathbb{R}$ *such that one of the following conditions*

(i) φ *is convex and* φ^{-1} *is operator monotone,*
(i') φ *is concave and* $-\varphi^{-1}$ *is operator monotone,*

is satisfied, where $\delta_\varphi = \varphi(\bar{m}) + \varphi(\bar{M}) - 2\varphi\left(\frac{\bar{m}+\bar{M}}{2}\right)$, $\widetilde{A}_{n_1} = \frac{1}{2}1_K - \frac{1}{\alpha(\bar{M}-\bar{m})} \times \sum_{i=1}^{n_1}\Phi_i\left(\left|A_i - \frac{\bar{m}+\bar{M}}{2}1_H\right|\right)$ *and* $\bar{m} \in [m, m_L]$, $\bar{M} \in [M_R, M]$, $\bar{m} < \bar{M}$*, are arbitrary numbers.*

But, if one of the following conditions

(ii) φ *is concave and* φ^{-1} *is operator monotone,*
(ii') φ *is convex and* $-\varphi^{-1}$ *is operator monotone,*

is satisfied, then the reverse inequality is valid in (3.8).

Proof. The inequalities (3.7) follows from Theorem 3.1 by replacing ψ with the identity function, while the inequalities (3.8) follows by replacing φ with the identity function and ψ with φ. □

Remark 3.3. *Let the assumptions of Theorem 3.1 be valid.*

1) We observe that if one of the following conditions

(i) $\psi \circ \varphi^{-1}$ is convex and ψ^{-1} is operator monotone,
(i') $\psi \circ \varphi^{-1}$ is concave and $-\psi^{-1}$ is operator monotone,

is satisfied, then the following obvious inequality (see Remark 2.4.1))

$$\mathcal{M}_\varphi(\beta, \mathbf{A}, \mathbf{\Phi}, n_1{+}1, n) \leq \psi^{-1}\left(\frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(\psi(A_i)) - \delta_\varphi\widetilde{A}_\beta\right) \leq \mathcal{M}_\psi(\beta, \mathbf{A}, \mathbf{\Phi}, n_1{+}1, n),$$

holds, $\delta_\varphi = \varphi(\bar{m})+\varphi(\bar{M})-2\varphi\left(\frac{\bar{m}+\bar{M}}{2}\right)$, $\widetilde{A}_\beta = \frac{1}{2}1_K - \frac{1}{\bar{M}-\bar{m}}\left|\frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i A_i - \frac{\bar{m}+\bar{M}}{2}1_K\right|$ and $\bar{m}\in[m,m_L]$, $\bar{M}\in[M_R,M]$, $\bar{m}<\bar{M}$, are arbitrary numbers.

2) We denote by m_φ and M_φ the bounds of $\mathcal{M}_\varphi(1,\mathbf{A},\mathbf{\Phi},1,n)$. If $(m_\varphi,M_\varphi)\cap[m_i,M_i]=\emptyset$, $i=1,\ldots,n_1$, and one of two following conditions

(i) $\psi\circ\varphi^{-1}$ is convex and ψ^{-1} is operator monotone

(ii) $\psi\circ\varphi^{-1}$ is concave and $-\psi^{-1}$ is operator monotone

is satisfied, then the double inequality (3.4) can be extended from the left side as follows

$$\begin{aligned}
&\mathcal{M}_\varphi(1,\mathbf{A},\mathbf{\Phi},1,n) = \mathcal{M}_\varphi(1,\mathbf{A},\mathbf{\Phi},1,n_1) \le \psi^{-1}\left(\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f(A_i)) - \delta_{\varphi,\psi}\widetilde{A}_\alpha\right)\\
\le\ & \mathcal{M}_\psi(\alpha,\mathbf{A},\mathbf{\Phi},1,n_1) \le \psi^{-1}\left(\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i\left(\psi(A_i)\right) + \beta\delta_{\varphi,\psi}\widetilde{A}_{\varphi,n_1,\alpha}\right) \le \mathcal{M}_\psi(1,\mathbf{A},\mathbf{\Phi},1,n)\\
\le\ & \psi^{-1}\left(\frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i\left(\psi(A_i)\right) - \alpha\delta_{\varphi,\psi}\widetilde{A}_{\varphi,n_1,\alpha}\right) \le \mathcal{M}_\psi(\beta,\mathbf{A},\mathbf{\Phi},n_1+1,n),
\end{aligned}$$

where $\delta_{\varphi,\psi}$ and $\widetilde{A}_{\varphi,n_1,\alpha}$ are defined by (3.2),

$$\widetilde{A}_\alpha = \frac{1}{2}1_K - \frac{1}{\bar{M}-\bar{m}}\left|\frac{1}{\alpha}\sum_{i=n_1+1}^{n}\Phi_i A_i - \frac{\bar{m}+\bar{M}}{2}1_K\right|.$$

As a special case of the quasi-arithmetic mean (3.1) we can study the weighted power mean as follows. For a subset $\{A_{p_1},\ldots,A_{p_2}\}$ of $\{A_1,\ldots,A_n\}$ we denote this mean by

$$M^{[r]}(\gamma,\mathbf{A},\mathbf{\Phi},p_1,p_2) = \begin{cases} \left(\dfrac{1}{\gamma}\displaystyle\sum_{i=p_1}^{p_2}\Phi_i\left(A_i^r\right)\right)^{1/r}, & r\in\mathbb{R}\backslash\{0\},\\ \exp\left(\dfrac{1}{\gamma}\displaystyle\sum_{i=p_1}^{p_2}\Phi_i\left(\ln\left(A_i\right)\right)\right), & r=0,\end{cases}$$

where $(A_{p_1},\ldots,A_{p_2})$ are strictly positive operators, $(\Phi_{p_1},\ldots,\Phi_{p_2})$ are positive linear mappings $\Phi_i:\mathcal{B}(H)\to\mathcal{B}(K)$ such that $\sum_{i=p_1}^{p_2}\Phi_i(1_H)=\gamma\,1_K$.

Under the same conditions, for convenience we introduce denotations as a special case of (3.2) as follows

$$\begin{aligned}
\delta_{r,s}(m,M) &= \begin{cases} m^s+M^s-2\left(\frac{m^r+M^r}{2}\right)^{s/r}, & r\neq 0,\\ m^s+M^s-2\left(mM\right)^{s/2}, & r=0,\end{cases}\\
\widetilde{A}_r(m,M) &= \begin{cases} \frac{1}{2}1_K - \frac{1}{|M^r-m^r|}\left|\sum_{i=1}^{n}\Phi_i(A_i^r) - \frac{M^r+m^r}{2}1_K\right|, & r\neq 0,\\ \frac{1}{2}1_K - \left|\ln\left(\frac{M}{m}\right)\right|^{-1}\left|\sum_{i=1}^{n}\Phi_i(\ln A_i) - \ln\sqrt{Mm}1_K\right|, & r=0,\end{cases}
\end{aligned} \tag{3.9}$$

where $m, M \in \mathbb{R}$, $0 < m < M$ and $r, s \in \mathbb{R}$, $r \leq s$. Of course, we include implicitly that $\widetilde{A}_r(m,M) \equiv \widetilde{A}_{r,A}(m,M)$, where $A = \sum_{i=1}^n \Phi_i(A_i^r)$ for $r \neq 0$ and $A = \sum_{i=1}^n \Phi_i(\ln A_i)$ for $r = 0$.

We obtain the following corollary by applying Theorem 3.1 to the above mean. This is an extension of [7, Corollary 13] and a refinement of [6, Corollary 3.4].

Corollary 3.4. *Let $(A_1, \ldots, A_n)$ be an $n-$tuple of self-adjoint operators $A_i \in B(H)$ with the bounds m_i and M_i, $m_i \leq M_i$, $i = 1, \ldots, n$. Let $(\Phi_1, \ldots, \Phi_n)$ be an $n-$tuple of positive linear mappings $\Phi_i : B(H) \to B(K)$, such that $\sum_{i=1}^{n_1} \Phi_i(1_H) = \alpha\, 1_K$, $\sum_{i=n_1+1}^{n} \Phi_i(1_H) = \beta\, 1_K$, where $1 \leq n_1 < n$, $\alpha, \beta > 0$ and $\alpha + \beta = 1$. Let*

$$(m_L, M_R) \cap [m_i, M_i] = \emptyset \quad \text{for} \quad i = n_1+1, \ldots, n, \qquad m < M,$$

where $m_L = \min\{m_1, \ldots, m_{n_1}\}$, $M_R = \max\{M_1, \ldots, M_{n_1}\}$ and

$$m = \begin{cases} m_L, & \text{if } \{M_i \colon M_i \leq m_L, i \in \{n_1+1, \ldots, n\}\} = \emptyset, \\ \max\{M_i \colon M_i \leq m_L, i \in \{n_1+1, \ldots, n\}\}, & \text{otherwise,} \end{cases}$$
$$M = \begin{cases} M_R, & \text{if } \{m_i \colon m_i \geq M_R, i \in \{n_1+1, \ldots, n\}\} = \emptyset, \\ \min\{m_i \colon m_i \geq M_R, i \in \{n_1+1, \ldots, n\}\}, & \text{otherwise.} \end{cases}$$

(i) *If either $r \leq s$, $s \geq 1$ or $r \leq s \leq -1$ and also one of two equalities*

$$\mathcal{M}^{[r]}(\alpha, \mathbf{A}, \mathbf{\Phi}, 1, n_1) = \mathcal{M}^{[r]}(1, \mathbf{A}, \mathbf{\Phi}, 1, n) = \mathcal{M}^{[r]}(\beta, \mathbf{A}, \mathbf{\Phi}, n_1+1, n)$$

is valid, then

$$\mathcal{M}^{[s]}(\alpha, \mathbf{A}, \mathbf{\Phi}, 1, n_1) \leq \left(\frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i\left(A_i^s\right) + \beta \delta_{r,s} \widetilde{A}_{s,n_1,\alpha} \right)^{1/s} \leq \mathcal{M}^{[s]}(1, \mathbf{A}, \mathbf{\Phi}, 1, n)$$
$$\leq \left(\frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i\left(A_i^s\right) - \alpha \delta_{r,s} \widetilde{A}_{s,n_1,\alpha} \right)^{1/s} \leq \mathcal{M}^{[s]}(\beta, \mathbf{A}, \mathbf{\Phi}, n_1+1, n)$$

holds, where $\delta_{r,s} \geq 0$ and $\widetilde{A}_{s,n_1,\alpha} \geq 0$.

In this case, we assume that $\delta_{r,s} \equiv \delta_{r,s}(\bar{m}, \bar{M})$, $\widetilde{A}_{s,n_1,\alpha} \equiv \widetilde{A}_{s,n_1,\alpha}(\bar{m}, \bar{M})$ are defined by (3.9) and $\bar{m} \in [m, m_L]$, $\bar{M} \in [M_R, M]$, $\bar{m} < \bar{M}$, are arbitrary numbers.

(ii) *If either $r \leq s$, $r \leq -1$ or $1 \leq r \leq s$ and also one of two equalities*

$$\mathcal{M}^{[s]}(\alpha, \mathbf{A}, \mathbf{\Phi}, 1, n_1) = \mathcal{M}^{[s]}(1, \mathbf{A}, \mathbf{\Phi}, 1, n) = \mathcal{M}^{[s]}(\beta, \mathbf{A}, \mathbf{\Phi}, n_1+1, n)$$

is valid, then

$$\mathcal{M}^{[r]}(\alpha, \mathbf{A}, \mathbf{\Phi}, 1, n_1) \geq \left(\frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i\left(A_i^r\right) + \beta \delta_{s,r} \widetilde{A}_{r,n_1,\alpha} \right)^{1/r} \geq \mathcal{M}^{[r]}(1, \mathbf{A}, \mathbf{\Phi}, 1, n)$$
$$\geq \left(\frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i\left(A_i^r\right) - \alpha \delta_{s,r} \widetilde{A}_{r,n_1,\alpha} \right)^{1/r} \geq \mathcal{M}^{[r]}(\beta, \mathbf{A}, \mathbf{\Phi}, n_1+1, n)$$

holds, where $\delta_{s,r} \leq 0$ and $\widetilde{A}_{s,n_1,\alpha} \geq 0$.

In this case, we assume that $\delta_{s,r} \equiv \delta_{s,r}(\bar{m}, \bar{M})$, $\widetilde{A}_{r,n_1,\alpha} \equiv \widetilde{A}_{r,n_1,\alpha}(\bar{m}, \bar{M})$ are defined by (3.9) and $\bar{m} \in [m, m_L]$, $\bar{M} \in [M_R, M]$, $\bar{m} < \bar{M}$, are arbitrary numbers.

Proof. In the case (i) we put $\psi(t) = t^s$ and $\varphi(t) = t^r$ if $r \neq 0$ or $\varphi(t) = \ln t$ if $r \neq 0$ in Theorem 3.1. In the case (ii) we put $\psi(t) = t^r$ and $\varphi(t) = t^s$ if $s \neq 0$ or $\varphi(t) = \ln t$ if $s \neq 0$. We omit the details. □

References

1. S. Abramovich, G. Jameson and G. Sinnamon, *Refining Jensens inequality*, Bull. Math. Soc. Sci. Math. Roumanie (N.S.) **47** 95 (2004), no. 1–2, 3-14.
2. S. S. Dragomir, *A new refinement of Jensen's inequality in linear spaces with applications*, Math. Comput. Modelling **52** (2010), 1497–1505.
3. F. Hansen, J. Pečarić and I. Perić, *Jensen's operator inequality and it's converses*, Math. Scand. **100** (2007), 61–73.
4. M. Khosravi, J. S. Aujla, S. S. Dragomir and M. S. Moslehian, *Refinements of Choi-Davis-Jensen's inequality*, Bull. Math. Anal. Appl. **3** 2 (2011), 127–133.
5. J. Mićić, Z. Pavić and J. Pečarić, *Jensen's inequality for operators without operator convexity*, Linear Algebra Appl. **434** (2011), 1228–1237.
6. J. Mićić, Z. Pavić and J. Pečarić, *Extension of Jensen's operator inequality for operators without operator convexity*, Abstr. Appl. Anal. **2011** (2011), 1–14.
7. J. Mićić, J. Pečarić and J. Perić, *Refined Jensen's operator inequality with condition on spectra*, Oper. Matrices, (2011), submitted for publication.
8. D.S. Mitrinović, J.E. Pečarić and A.M. Fink, Classical and New Inequalities in Analysis, Kluwer Acad. Publ., Dordrecht-Boston-London, 1993.
9. B. Mond and J. Pečarić, *Converses of Jensen's inequality for several operators*, Revue d'analyse numer. et de théorie de l'approxim. **23** (1994) 179–183.
10. M. S. Moslehian, *Operator extensions of Huas inequality*, Linear Algebra Appl. **430** (2009), 1131–1139.
11. J. Rooin, *A refinement of Jensens inequality*, J. Ineq. Pure and Appl. Math., **6**, 2 (2005), Art. 38., 4 pp.
12. H. M. Srivastava, Z.-G. Xia, and Z.-H. Zhang, *Some further refinements and extensions of the HermiteHadamard and Jensen inequalities in several variables*, Math. Comput. Modelling **54** (2011), 2709–2717.
13. Z.-G. Xiao, H. M. Srivastava and Z.-H. Zhang, *Further refinements of the Jensen inequalities based upon samples with repetitions*, Math. Comput. Modelling **51** (2010), 592–600.
14. L.-C. Wang, X.-F. Ma and L.-H. Liu, *A note on some new refinements of Jensens inequality for convex functions*, J. Inequal. Pure Appl. Math., **10**, 2 (2009), Art. 48., 6 pp.

[1] Faculty of Mechanical Engineering and Naval Architecture, University of Zagreb, Ivana Lučića 5, 10000 Zagreb, Croatia.
E-mail address: jmicic@fsb.hr

[2] Faculty of Textile Technology, University of Zagreb, Prilaz baruna Filipovića 30, 10000 Zagreb, Croatia.
E-mail address: pecaric@hazu.hr

[3] Faculty of Science, Department of Mathematics, University of Split, Teslina 12, 21000 Split, Croatia.
E-mail address: jperic@pmfst.hr