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# REVIEW OF GAMMA FUNCTIONS IN ACCUMULATED FATIGUE DAMAGE ASSESSMENT OF SHIP STRUCTURES


## Abstract

Palmgren-Miner (P-M) approach is the oldest, simplest and the most frequently applied method for cumulative fatigue damage assessment of ship structures. By assuming Weibull distribution as probability distribution of stress ranges, closed-form P-M expressions for accumulated damage calculation are adopted by Ship Classification Societies. There are several versions of such expressions currently in use and all of them implement various kinds of Gamma functions. Accurate evaluation of Gamma functions is therefore of crucial importance for reliable accumulated damage calculation. However, it is noticed that fatigue rules provides insufficient attention to the precise definition of Gamma functions, causing unnecessary wasting of time and possible mistakes. The present paper gives review of expressions for accumulated damage assessment proposed by major Classification Societies aiming to clarify which kind of Gamma functions are used and how to calculate them efficiently.


*Key words: fatigue damage, gamma functions, classification societies*

# PREGLED GAMMA FUNKCIJA KOD AKUMULIRANOG PRORAČUNA ZAMORA BRODSKIH KONSTRUKCIJA


## Sažetak

Palmgren-Miner (P-M) pristup je najstarija, najjednostavnija i najčešće korištena metoda za proračun zamornog oštećenja brodskih konstrukcija. Pod pretpostavkom Weibullove distribucije raspona naprezanja, klasifikacijska društva su usvojila P-M analitičke izraze za proračun akumuliranog oštećenja. U upotrebi postoji više verzija sličnih izraza te svaki od njih uključuje neki oblik Gamma funkcija. Stoga, točna procjena Gamma funkcija predstavlja jedan od ključnih parametara za pouzdan proračun zamornog oštećenja. Primijećeno je da pravila za proračun zamora kod brodskih konstrukcija ne posvećuju dovoljno pažnje preciznoj definiciji Gamma funkcija, uzrokujući tako gubitak vremena i eventualne greške u proračunu. Ovaj članak predstavlja pregled izraza za proračun akumuliranog oštećenja važnijih klasifikacijskih društava, nastojeći rasvijetliti kako i koji tip Gamma funkcije efikasno odrediti.


*Ključne riječi: zamorno oštećenje, gamma funkcije, klasifikacijska društva*

## 1. Introduction

All major Classification Societies prescribe closed-form expressions for the fatigue damage assessment. Usually, such procedure requires calculation of stress ranges, selection of the appropriate S-N curve and finally, calculation of the cumulative damage. Although all Classification Societies base their fatigue damage formulations on the Palmgren-Miner (P-M) approach, at the same time they prescribe rather different closed-form expressions. These expressions mostly differ in implementation of various Gamma functions and therefore accurate evaluation of Gamma functions is one of crucial aspects for reliable accumulated damage calculation. However, it is noticed that fatigue rules prescribe various Gamma functions in the form of tables. Since, Gamma functions grow so rapidly, the probability of inaccurate readings of proposed tables is large which may cause unnecessary waste of time and possible mistakes.

Present paper gives an overview of Gamma functions and possible numerical calculation in the computing environments, i.e. spreadsheets. Furthermore the focus of the present paper is to investigate the influence of various Gamma functions on the fatigue damage and to check whether different closed-form expressions provide the same results if the same stress level and the same S-N curve are assumed in all formulations.

## 2. Gamma function

The theory of the gamma function was developed in connection with the problem of generalizing the factorial function of the natural numbers, that is, the problem of finding an expression that has the value n! for positive integers n, and that can be extended to arbitrary real numbers at the same time. So, the gamma function can be seen as a solution for graphical interpolation of the factorial function to non-integer values as it is presented in **Fig. 1**.

Various eminent mathematicians, starting from Euler had been dealing with this problem. However, Gauss rewrote Euler's integral definition, proved several theorems regarding factorial functions and first defined the factorial function, originally known as the pi function:

$$\Pi(n) = n! \tag{1}$$

Later on Legendre also rewrote Euler's integral definition and normalized factorial function to its modern form known as the Gamma function:

$$\Gamma(n+1) = n! \tag{2}$$

Therefore, usually used complete Gamma function can be consider as an extension of the factorial function, with its argument shifted down by 1, to real and complex numbers. For a complex number x with positive real part ($\mathrm{Re}[x]>0$), the Gamma function is defined via a convergent improper integral as:

$$\Gamma(x) = \int_0^\infty t^{x-1} \cdot e^{-t} dt \tag{3}$$

This definition can be extended by analytic continuation to the rest of the complex plane, except the non-positive integers and zero, as it is presented in **Fig. 1**. Integrating equation (3) by parts for a real argument, it can be seen that:

$$\Gamma(x) = \int_0^\infty t^{x-1} \cdot e^{-t} dt = \left[-t^{x-1} \cdot e^{-t}\right]\Big|_0^\infty + \int_0^\infty (x-1)t^{x-2} \cdot e^{-t} dt = (x-1)\Gamma(x-1) \tag{4}$$

If x is a positive integer n, then the Gamma function reduces to the factorial function as:

$$\Gamma(n)=(n-1)! \tag{5}$$

Furthermore, the complete Gamma function $\Gamma(x)$ can be generalized to the upper incomplete gamma function $\Gamma(a,x)$ and lower incomplete gamma function $\gamma(a,x)$. In the integral in Equation (3), which defines the Gamma function, the limits of integration are fixed. The upper and lower incomplete Gamma functions are the functions obtained by allowing the lower or upper (respectively) limit of integration to vary. Thus, the upper Gamma function may be represented as:

$$\Gamma(a,x)=\int_x^\infty t^{a-1}\cdot e^{-t}dt \tag{6}$$

while the lower Gamma function can be represented as:

$$\gamma(a,x)=\int_0^x t^{a-1}\cdot e^{-t}dt \tag{7}$$

By definition, for all positive real numbers x, the lower and upper incomplete gamma functions must satisfy following equation:

$$\Gamma(a,x)+\gamma(a,x)=\Gamma(a) \tag{8}$$

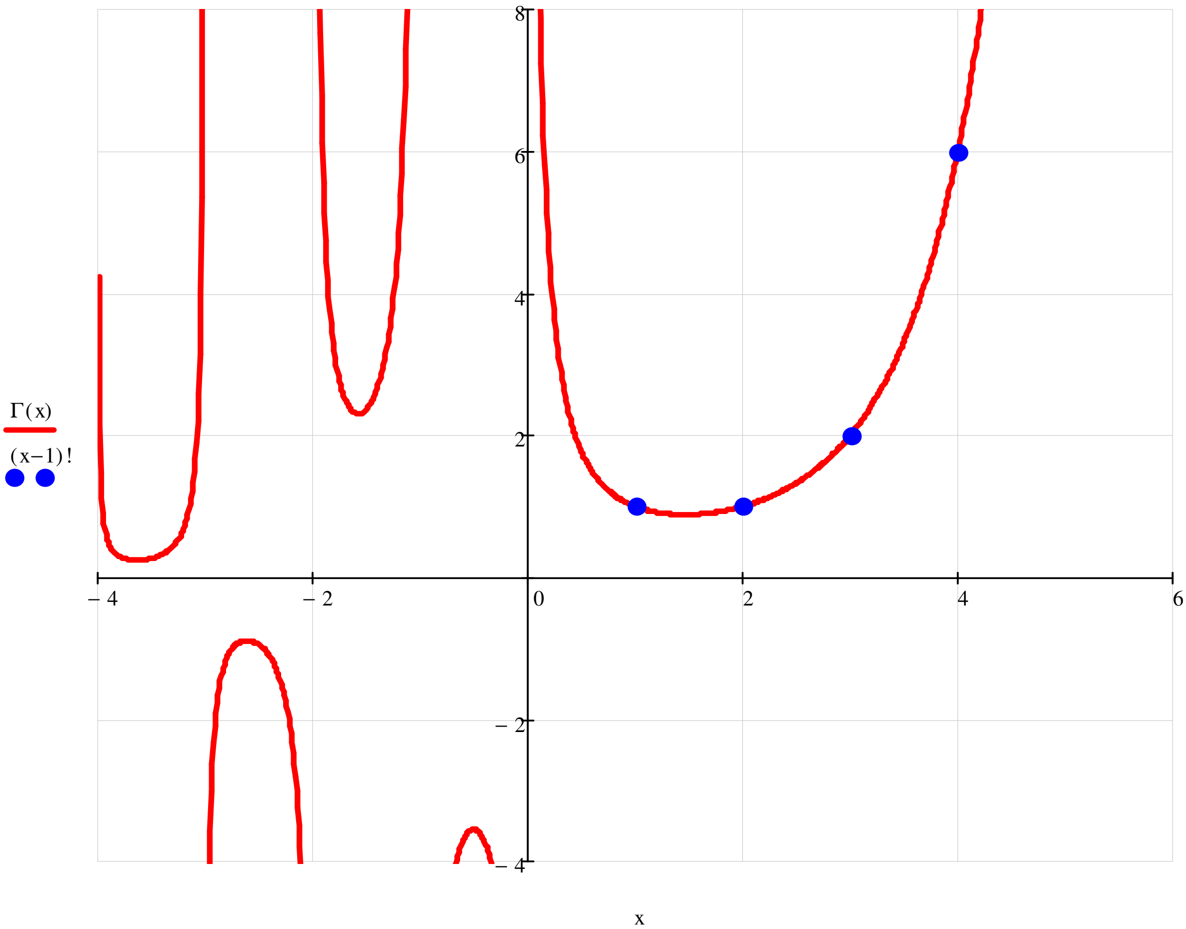


**Fig. 1.** Gamma function and the approximation to the factorial function

Because the Gamma and factorial functions grow so rapidly for moderately-large arguments, many computing environments include a function that returns the natural logarithm of the Gamma function (often given the name lngamma in programming environments or gammaln in spreadsheets). Such functions grow much more slowly, and for combinatorial calculations allow adding and subtracting logs instead of multiplying and dividing very large values. Thus, if the complete Gamma function is to be calculated, for instance in the Excel spreadsheet, one should use following expression:

$$\Gamma(a) = e^{gamma\ln(a)} \tag{9}$$

The upper Gamma function can be calculated by employing Equation (8) where the lower Gamma function in the Excel spreadsheet can be calculated by employing expression:

$$\gamma(a,x) = e^{gamma\ln(a)} \cdot gamma.dist(x,a,1,1) \tag{10}$$

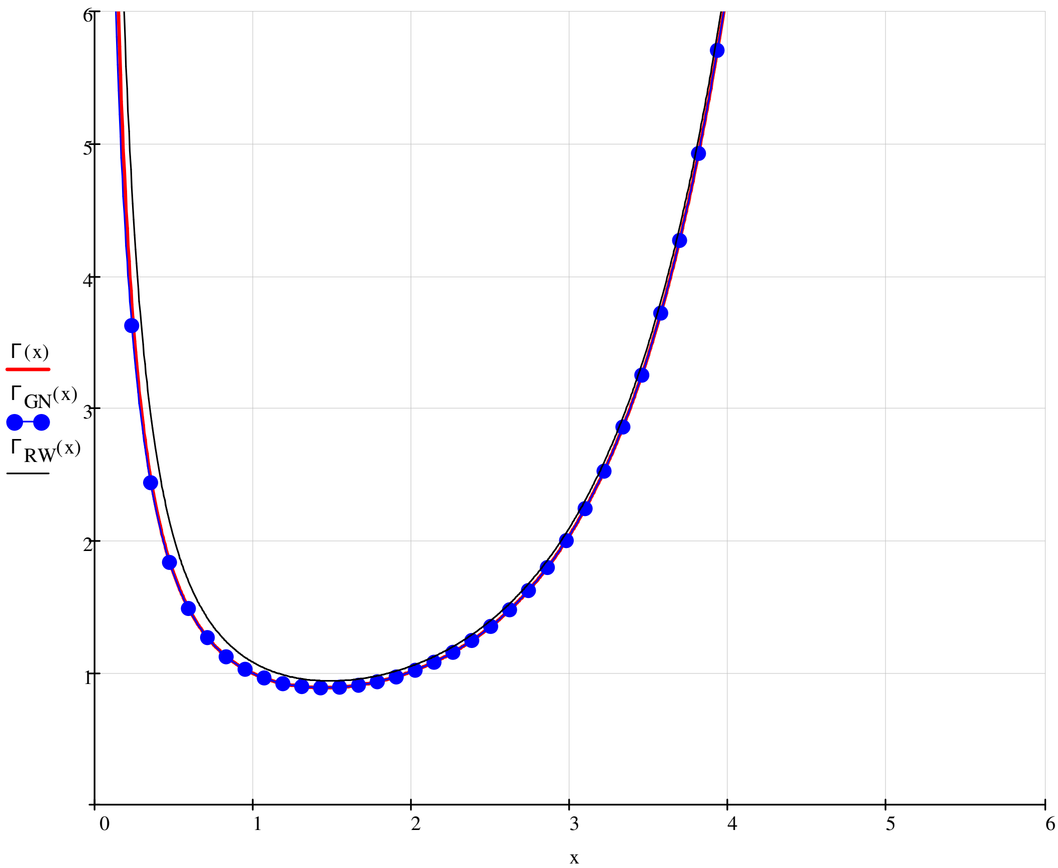


**Fig. 2.** Comparison of approximate Gamma function formulations with the complete Gamma function

Besides programming environments and spreadsheets the complete Gamma function can also be calculated by approximation as proposed by Robert Windschitl in 2002:

$$\Gamma_{RW}(a) = \sqrt{\frac{2\cdot\pi}{a}}\left[\frac{a}{e}\left(n\cdot\sinh\frac{1}{a}+\frac{1}{810\cdot a^6}\right)\right]^a \tag{11}$$

or slightly simpler approximation as proposed by Gergo Nemes in 2007:

$$\Gamma_{GN}(a) = \sqrt{\frac{2\cdot\pi}{a}}\left[\frac{1}{e}\left(a+\frac{1}{12\cdot a-\frac{1}{10\cdot a}}\right)\right]^{a} \tag{12}$$

Comparison of approximate Gamma function formulations with the complete Gamma function for positive real numbers is given in Fig. **2**. It can be concluded that approximation as proposed by Robert Windschitl slightly deviates from the complete Gamma function for real numbers below 2, while approximate formulation, as proposed by Gergo Nemes agrees perfectly with the complete Gamma function for the entire domain of positive real numbers.

The Gamma function is a component in various probability-distribution functions, and as such it is applicable in the fields of probability and statistics, as well as combinatorics. Within the field of naval architecture it is usually used in the fatigue analysis of ships and offshore structures, which is elaborated in next section.

## 3. Fatigue damage

Assessment of the fatigue strength of welded steel structural members includes the following three crucial phases:

(a) Calculation of stress ranges

(b) Selection of the design S-N curve

(c) Calculation of the cumulative damage.

The calculation of stress ranges can be based on the simplified approach or direct spectral analysis which includes calculation of load transfer functions. Classification Societies Rules usually prescribe simplified expressions for various global and local loads which are used for determination of the stress ranges for all relevant loading conditions, while direct spectral analysis is only required in some special cases.

Next important phase in the assessment of the fatigue capacity is the careful selection of the design S-N curve. These curves are usually prescribed by Classification Societies in the form of classes as it is presented in the **Table 8**, where they define the fatigue life of welded joints and flame-cut edges as a number of constant amplitude load cycles N until failure. Most of the S-N curves that give the relationship between the fatigue life and the nominal stress range are determined in laboratories for specific local weld detail and does not include the geometry of the structural detail. Stress concentrations due to the specific detail geometry and possible notch effects have to be taken into account through the stress concentration factors when calculating stress range. Irrespective of the type of stress approach considered, the selected design S-N curve has to be modified to take into account that experimental S-N curves are generally mean curves determined for constant amplitude load cycles. Therefore, experimental S-N curves are usually defined by their mean fatigue life and standard deviation where the mean S-N curve gives the stress level S at which the structural detail will fail with a probability level of 50% after N loading cycles. Moreover, the basic design S-N curve have to be corrected to take into account several effects which are not properly taken into account in the experimental S-N curves. However, various influences on the design S-N curve, such as influence of residual stresses, thickness, material etc. are neglected in the present paper.

**Table 8.** Basic S-N curve data, in-air from Common Structural Rules for double hull oil tankers [1]

| Class | $K_1$ | | | $m_1$ | σ Standard deviation | | $K_2$ | $S_q$ N/mm$^2$ | $m_2$ |
|---|---|---|---|---|---|---|---|---|---|
| | | $\log_{10}$ | $\log_e$ | | $\log_{10}$ | $\log_e$ | | | |
| B | 2.343E+15 | 15.3697 | 35.39 | 4 | 0.1821 | 0.4194 | 1.01E+15 | 100.2 | 7 |
| C | 1.082E+14 | 14.0342 | 32.3153 | 3.5 | 0.2041 | 0.47 | 4.23E+13 | 78.2 | 6.5 |
| D | 3.988E+12 | 12.6007 | 29.0144 | 3 | 0.2095 | 0.4824 | 1.52E+12 | 53.4 | 5 |
| E | 3.289E+12 | 12.5169 | 28.8216 | 3 | 0.2509 | 0.5777 | 1.04E+12 | 47.0 | 5 |
| F | 1.726E+12 | 12.237 | 28.177 | 3 | 0.2183 | 0.5027 | 6.30E+11 | 39.8 | 5 |
| F2 | 1.231E+12 | 12.09 | 27.8387 | 3 | 0.2279 | 0.5248 | 4.30E+11 | 35.0 | 5 |
| G | 5.66E+11 | 11.7525 | 27.0614 | 3 | 0.1793 | 0.4129 | 2.50E+11 | 29.2 | 5 |
| W | 3.68E+11 | 11.5662 | 26.6324 | 3 | 0.1846 | 0.4251 | 1.60E+11 | 25.2 | 5 |

The third phase in fatigue analysis is calculation of the cumulative damage. Palmgren-Miner (P-M) approach is the oldest, simplest and the most frequently applied method for cumulative fatigue damage assessment of ship structures. According to the P-M rule, the total damage the structure is experiencing may be expressed as the accumulated damage from each load cycle at different stress levels, independent of the sequence in which the stress cycles occur. If the damage contributed by one cycle of stress range $S_i$ is $1/N_i$, where $N_i$ is the mean fatigue life under a constant amplitude stress range $S_i$ obtained from the S-N curve, the cumulative damage D caused by stress range $S_1$, $S_2$, $S_n$ applied $n_1$, $n_2$, $n_k$ may be obtained by superposition as:

$$D = \sum_{i=1}^{i=k} \frac{n_i}{N_i} \tag{13}$$

The structure is considered as failed when the cumulative damage ratio D is equal to unity or greater.

Based on Palmgren-Miner approach and by assuming Weibull distribution as probability distribution of stress ranges, different Classification Societies proposed different closed-form P-M expressions for accumulated damage calculation. Since all expressions are based on the same stress probability distribution, they mostly differ in implementation of various kinds Gamma functions. Since it is noticed that fatigue rules provides insufficient attention to the precise definition of Gamma functions, the focus of the present paper is to investigate the influence of various Gamma functions on the fatigue damage if the same stress level and the same S-N curve for all expressions are assumed.

### 3.1. Damage expression according Bureau Veritas Rules, Common Structural Rules (CSR) for Oil Tankers and Harmonized Common Structural Rules

After detailed inspection of Bureau Veritas Rules, Common Structural Rules (CSR) for Oil Tankers and Harmonized Common Structural Rules it is concluded that all three considered Rules provide the same closed-form expression for fatigue damage assessment [1][2][3]. The only difference within formulations is in the prescribed S-N curves, but this is not in the focus of the present paper.

Therefore, as a representative expression for the cumulative damage ratio with the two slope S-N curve, the Common Structural Rules formulation is assumed [1]:

$$DM_i = \frac{\alpha_i N_L}{K_2} \cdot \frac{S_{Ri}{}^m}{\left(\ln N_R\right)^{m/\xi}} \cdot \mu_i \cdot \Gamma\left(\frac{m}{\xi}+1\right) \tag{14}$$

where $N_L$ is the number of cycles for the expected design life. Unless stated otherwise, $N_L$ to be taken as:

$$N_L = \frac{f_0 U}{4 \log L} \tag{15}$$

Where $f_0 = 0.85$, is the factor taking into account non-sailing time for operations such as loading and unloading, repairs, etc. Design life, $U = 0.788 \cdot 10^9$ is given in seconds for a design life of 25 years, while $L$ is the rule length in meters. S-N curve parameters $m$ and $K_2$ are defined in the **Table 8**, while $\alpha_i$ is the proportion of the ship's life for specific loading condition. Stress range in N/mm$^2$ $S_{Ri}$ and number of stress cycles $N_R$ correspond to the representative probability level of $10^{-4}$, while $\xi$ is the Weibull distribution parameter. Gamma function has to be defined based on S-N curve parameters $m$ and Weibull distribution parameter $\xi$. Change in slope of the S-N curve is taken into account through the coefficient $\mu_i$ as:

$$\mu_i = 1 - \frac{\left(\gamma\left(\frac{m}{\xi}+1, \nu_i\right) - \nu_i^{-\Delta m/\xi} \cdot \gamma\left(\frac{m+\Delta m}{\xi}+1, \nu_i\right)\right)}{\Gamma\left(\frac{m}{\xi}+1\right)} \tag{16}$$

where $\Delta m$ is the slope change of the upper to lower segment of the S-N curve, $\gamma(a,x)$ is the incomplete Gamma function (Equation (10)), while coefficient $\nu_i$ depends on the stress range at the intersection of the two segments of the S-N curve $S_q$ as:

$$\nu_i = \left(\frac{S_q}{S_{Ri}}\right)^{\xi} \ln N_R \tag{17}$$

### 3.2. Damage expression according Det Norske Veritas Rules

When a bi-linear or two-slope S-N curve is used, the fatigue damage expression is given according to the DNV classification notes as [4]:

$$D = \nu_0 \cdot T_d \left[\frac{q^{m_1}}{a_1} \cdot \Gamma\left(\frac{m_1}{h}+1, \left(\frac{S_1}{q}\right)^h\right) + \frac{q^{m_2}}{a_2} \cdot \gamma\left(\frac{m_2}{h}+1, \left(\frac{S_1}{q}\right)^h\right)\right] \tag{18}$$

Where $\nu_0$ is the long-term average response zero-crossing frequency, $T_d$ is design life of ship in seconds. The Weibull scale parameter is defined from the stress range level, $\Delta\sigma_0$, out of $n_0$ cycles as:

$$q = \frac{\Delta\sigma_0}{\left(\ln n_0\right)^{1/h}} \tag{19}$$

where $h$ is the Weibull shape parameter. Stress range $S_1$ for which change of slope of S-N curve occur, corresponds to the stress range $S_q$ from **Table 8**. These parameters are needed for calculation of the complementary incomplete Gamma function $\Gamma(\ ,\ )$(Equation (8)) and incomplete Gamma function $\gamma(\ ,)$ (Equation (10)). Other S-N curve parameters $m_1, m_2, \overline{a_1}, \overline{a_2}$ also can be obtained from **Table 8** as:

$$\begin{aligned} \log \overline{a_1} &= \log K_1 - 2\sigma = \log K_2 \\ \log \overline{a_2} &= \log\left(N \cdot S_q^{m_2}\right) \end{aligned} \tag{20}$$

3.3. Damage expression according Common Structural Rules for Bulk Carriers

The elementary fatigue damage for each loading condition is to be calculated with the following formula [5]:

$$D_j = \frac{\alpha_j N_L}{K} \cdot \frac{{\Delta\sigma_{E,j}}^4}{\left(\ln N_R\right)^{4/\xi}} \cdot \left[\Gamma\left(\frac{4}{\xi}+1,\nu\right) + \nu^{-3/\xi} \cdot \gamma\left(\frac{7}{\xi}+1,\nu\right)\right] \tag{21}$$

In the Common structural rules for bulk carriers the S-N curve is already embedded in the expression for fatigue damage, since S-N curve parameter is specified as $K = 1.014 \cdot 10^{15}$. Furthermore, $N_L$ corresponds to the Equation (15) from Bureau Veritas rules, while parameters $\Delta\sigma_{E,j}$, $N_R$, $\xi$ correspond to the stress range in N/mm$^2$, number of stress cycles and the Weibull distribution parameter respectively. Coefficient $\alpha_j$ depends on the loading condition and is usually taken as 1 for the assessment of hatch corners, while parameter $\nu$ depends on the proposed S-N curve and is given as:

$$\nu = \left(\frac{100.3}{\Delta\sigma_{E,j}}\right)^{\xi} \ln N_R \tag{22}$$

The complementary incomplete Gamma function (Type 2 incomplete gamma function, $\Gamma(\ ,\ )$) can be calculated by employing Equation (8) and incomplete Gamma function (Type 1 incomplete gamma function, $\gamma(\ ,)$) can be calculated by employing Equation (10).

## 4. Example

In order to investigate the influence of the various Gamma functions on the fatigue damage, the same stress range of 300 N/mm$^2$ for 10000 number of stress cycles is assumed in all considered expressions given by Classification Societies. Since damage expression according Common structural rules for bulk carriers already implicitly includes the S-N curve, the same S-N curve should be employed for two other expressions as well. Therefore, the class B S-N curve from **Table 8** (**Fig. 3**) is employed for the Det Norske Veritas and Bureau Veritas (CSR Oil Tankers and Harmonized CSR) fatigue damage expression. Results of the assumed fatigue damage are presented in **Table 9**.

**Table 9** Fatigue damage according different Classification Societies expressions

| | **BV Rules CSR Oil Tankers Harmonized CSR** | **DNV Rules** | **CSR Bulk Carriers** |
|---|---|---|---|
| Stress range | $S_{Ri} = 300$ N/mm$^2$ | $\Delta\sigma_0 = 300$ N/mm$^2$ | $\Delta\sigma_{E,j} = 300$ N/mm$^2$ |
| Number of stress cycles | $N_R = 10000$ | $n_0 = 10000$ | $N_R = 10000$ |
| Total number of cycles for the design ship's life (25 years) | $N_L = 6.652 \cdot 10^7$ | $\nu_0 \cdot T_d = 6.652 \cdot 10^7$ | $N_L = 6.652 \cdot 10^7$ |
| Weibull distribution parameter | $\xi = 1$ | $h = 1$ | $\xi = 1$ |
| S-N curve parameters | $K_2 = 1.01 \cdot 10^{15}$<br>$m = 4$<br>$\Delta m = 3$<br>$S_q = 100.2$ N/mm$^2$ | $\overline{a_1} = 1.012 \cdot 10^{15}$<br>$\overline{a_2} = 1.022 \cdot 10^{21}$<br>$m_1 = 4$<br>$m_2 = 7$<br>$S_1 = 100.2$ N/mm$^2$ | $K = 1.014 \cdot 10^{15}$ |
| Fatigue damage | $DM_i = 0.593$ | $D = 0.593$ | $D_j = 0.592$ |

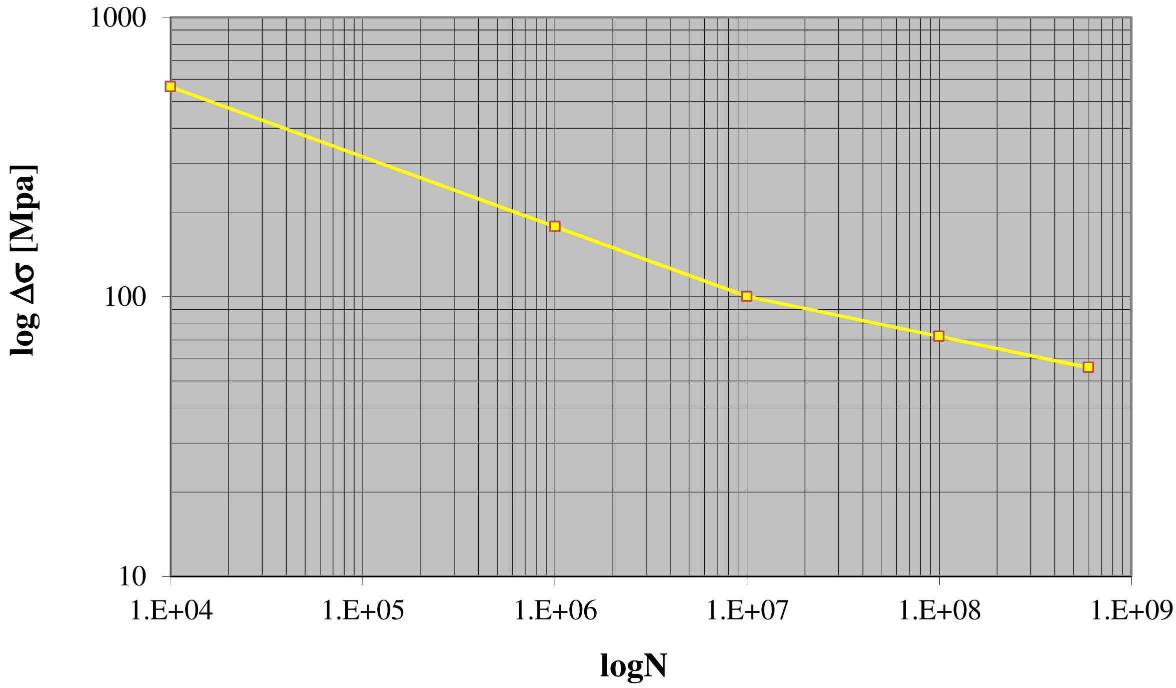


**Fig. 3.** S-N curve, Class B

## 5. Conclusion

Paper gives an overview of Gamma functions and their influence on the fatigue damage closed-form expressions provided by different Classification Societies. Five different Rules are investigated.

First, it is concluded that Bureau Veritas Rules, CSR for Oil Tankers and Harmonized CSR provide the same formulations for the fatigue damage assessment. Furthermore it is noticed that formulations of Det Norske Veritas Rules and CSR for Bulk Carriers mostly differ in application of different kinds Gamma functions. Besides that, all expressions enable application of various kinds of S-N curves except Harmonized CSR, which implicitly include the B class S-N curve within the fatigue damage formulation. Thus, the stated S-N curve in conjunction with same stress range is used to investigate the influence of the various Gamma functions on the fatigue damage formulations. It is concluded, as expected, that all expressions yield the same results if the Gamma functions are calculated correctly. Therefore, it is of great interest to determine Gamma functions numerically rather than use prescribed tables which could lead to unnecessary waste of time and possible mistakes.

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